An Approximate Expression for Laminar Skin-Friction Coefficient

A. Pozzi*
University of Naples, Naples, Italy

1. Introduction

THE advent of electronic computers and their acceptance in industry and universities has changed markedly the course of research in serling approximate methods for solving technical problems. In fact, only simple methods are useful for giving approximate solutions. For the boundary-layer equations, the most important technique employed has been the integral method suggested by von Kármán and modified by many authors; among these, the recent work of Kosson[1] is interesting. He developed an idea of vonKármán and Millikan[2] by using the integral method only for the inner region of the boundary layer, while solving the outer one by means of a linearized form of the equation in the von Mises coordinates.

Another important analysis has been recently made by Gortler.[3] This proposes an accurate expansion in series for which he has published tables of the related universal functions. This series involves a parameter

$$\beta = 2u'\int_0^x \frac{Cdx}{C^2}$$

which is difficult to use in the general case. These procedures, as Kosson states "occupy a middle ground between the integral method and numerical methods."

It would be interesting now to use a different method that would lead to a simple direct expression that, though less accurate, would give by slide rule only, the value of the skin-friction coefficient. Two classes of outer velocities will be analyzed exhaustively: $U_e = x^a$ and

$$U_e^2 = \sum_{i=0}^{n} A_i x^{r_i}$$

The second case is general enough because it is simple to expand a function in a Taylor series or Legendre polynomials.

2. Boundary-Layer Equations

The incompressible two-dimensional boundary-layer equations in a nondimensional form in the von Mises coordinates (defined as $\xi = x$ and $\eta = \psi$, where ψ is the nondimensional stream function, i.e., $\psi_x = -v$ and $\psi_y = u$) lead to

$$Z_\xi = uZ_{\eta\eta} \tag{1}$$

where $Z = U_e^2 - u^2$ and U_e is the outer velocity.

A remarkable advantage of this transformation is that u_y, and hence the skin-friction coefficient proportional to it, is simply related to Z; one has, in fact $Z_\eta = -2\, u_y$.

Equation (1) is a parabolic type and will be solved in the positive quadrant; the boundary conditions are

$$Z(0, \eta) = 0 \tag{2}$$

$$Z(\xi, 0) = U_e^2 \tag{3}$$

The greatest difficulty in solving this problem is the nonlinearity of Eq. (1). The approximation suggested here consists of linearizing it by substituting the coefficient of $Z_{\eta\eta}$ in Eq. (1), "u" with an approximate expression. On studying the characteristics of such a function, one notes that the approximation must be better near the wall than for high values of η because $Z_{\eta\eta}$ vanishes quickly as η approaches infinity, and hence its coefficient is not too important. Moreover the behavior of u for small values of η is not analytic (the Jacobian of the transformation at $\eta = 0$ vanishes) but u^2 behaves analytically; in fact $Z_\eta(= -2u_y)$ on the wall cannot be infinity. It follows that the simplest expression satisfying these conditions near the wall is $u_W = \eta^{1/2}/f(\xi)$ and in the entire field $u = Cu_W$.

Received December 11, 1965; revision received May 19, 1965.
* Associate Professor of Aerodynamics.

3. Solution of Approximate Equation

It is possible to determine a posteriori the function f by means of the following transformation $d\xi/dz = f/C$. Then Eq. (1) becomes $Z_z = \eta^{1/2}Z_{\eta\eta}$ and can be solved in two ways.

Similar solutions

By putting $Z = \xi^r g(s)$ with $s = \eta^{3/2}/z$, one obtains the following equation for g: $9s^2g'' + (3 + 12r + 4s)sg' + 4r(r-1)g = 0$ whose exact solution can be given in term of confluent hypergeometric functions ϕ by

$$Z = Az^{2r/3}\exp(-4s/9)\,\{\phi[(1+2r)/3, \tfrac{1}{3}, (4s/g)] - Bs^{2/3}\phi[1 + (2r/3), \tfrac{5}{3}, 4/9s)]\} \tag{4}$$

where

$$\beta = (4/9)^{2/3}\Gamma(\tfrac{1}{3})\Gamma(2r/3 + 1)/\Gamma(\tfrac{5}{3})\Gamma[(2r+1)/3]$$

One of the two free constants B has been determined by the condition that Z vanishes at η = infinity; the second constant A depends on outer velocity according to Eq. (3). For the flat plate, it is $r = 0$, and Eq. (4) can be written in terms of the gamma function. Thus, $\Gamma(\tfrac{2}{3}, 4s/9)/\Gamma(\tfrac{2}{3})$.

Solutions by means of Laplace transformation

By introducing the Laplace transform of Z, i.e., putting $F(t, \eta) = L_z[Z(t, \eta)]$, the approximate equation becomes $\eta^{1/2}F'' - tF = 0$; this equation has as its solution, convergent at infinity, $F = D(t)\eta^{1/2}K_{2/3}(4t^{1/2}\eta^{3/4}/3)$ in which K is the modified Bessel function of the second kind, and by transforming Eq. (3) the following expression is obtained for D:

$$D = 2t^{1/3}(\tfrac{2}{3})^{2/3}L_z[U^2]/\Gamma(\tfrac{2}{3})$$

Therefore the transform of $Z\eta$, F' at $\eta = 0$ is given by:

$$F'(0) = t^{2/3}\Gamma(\tfrac{1}{3})(\tfrac{2}{3})^{1/3}L_z[U_e^2]/\Gamma(\tfrac{2}{3}) \tag{5}$$

4. Determination of the Function f

To determine the function f, it is assumed that the approximate expression for $u_W(= \eta^{1/2}/f)$ and the solution of $Z_z = \eta^{1/2}Z_{\eta\eta}$ give the same value for u_y on the wall; thus one has $f^{-2} = -Z_{\eta,0}$.

Similar solutions

In this case, it is simple to write f in terms of z; in fact Eq. (4) leads to the following expression:

$$f^{-2} = \{A(\tfrac{4}{9})^{2/3}\Gamma(\tfrac{1}{3})\Gamma(2T/3+1)/\Gamma(\tfrac{5}{3})\Gamma[(2T+1)/3]\}z^{2(T-1)/3} \tag{6}$$

Solutions by means of Laplace transformation

The function f, taking into account Eq. (5) and the convolution theorem, can be written as:

$$f^{-2} = \frac{(\tfrac{2}{3})^{1/3}}{\Gamma(\tfrac{2}{3})}\frac{d}{dz}\int_0^z (z-t)^{-2/3}U_e^2dt \tag{7}$$

Table 1 $u_y a^{-1/2}$ for $U_e = 1 - ax$

ax	Kosson	Howarth	Present solution
0.025	1.72	1.77	1.78
0.05	0.97	1.00	0.99
0.075	0.64	0.61	0.58
0.1	0.34	0.31	0.31

Table 2 Nondimensional shear stress

	Sinusinoidal velocity			Hiemenz distribution		
	Kosson	Exact	Present solution	Kosson	Exact	Present solution
0	0	0	0	1.22	1.23	1.25
30°	1.62	1.64	1.61	1.17	1.18	1.16
60°	2.22	2.26	2.02	D.90	0.92	0.70
90°	1.26	1.35	0.70	...	...	...
Separation	102°	109°	98°	78°	80°	78°

This equation can be considered as an "Abel equation"[4] whose known function is

$$\Gamma\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{-1/3}\int_0^z f^{-2}dz$$

Hence Eq. (7) can be written as

$$U_e^2 = \Gamma^{-1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{-1/3}\int_0^z (z-t)^{1/3}f^{-2}dt$$

This equation also is considered as an Abel equation, and one has the following simpler expression for f:

$$f^{-2} = \left[\frac{(\frac{2}{3})^{1/3}}{\Gamma(\frac{2}{3})}\right]\left[z^{-2/3}U_e^2(0) + \int_0^z (z-t)^{-2/3}(U_e^2)'dt\right] \quad (8)$$

Equation (8) can be solved either by an iterative technique or in an indirect way, assuming for $U_e(z)$ an arbitrary expression and determining the function $U_e(\xi)$ a posteriori. The constant C has been determined by equating the integrals between 0 and η^+ of the approximate expression assumed for Z, i.e., $1 - C^2f^{-2}\eta$, and that of the solutions already obtained; η^+ is the value at which Z vanishes, and it is f^2/C in the first case and infinity in the second. Such an evaluation, made in the case of the flat plate, leads to the following value for C: $C = \Gamma(\frac{2}{3})[3\Gamma(\frac{4}{3})]^{-1/2} = 0.827$.

5. Applications of the Method

Two classes of outer velocity will be considered: U_e given 1) by any power "a" of $x(=\xi)$, and 2) by functions, given by a linear combination of any power of x.

a) $U_e = kx^a$

From Eqs. (4) and (6) we have

$$z = \left[\frac{4CB^{1/2}k}{3(1+a)}\right]^{3/4} x^{3(1+a)/4}$$

$$B = \left(\frac{4}{9}\right)^{2/3} \times \frac{\Gamma(\frac{1}{3})\Gamma[(11a+3)/(3+3a)]}{\Gamma(\frac{5}{3})\Gamma[(3a+\frac{1}{3})/(1+a)]}$$

$$Z = k^2x^{2a}\exp\left(\frac{-4s}{9}\right)\left[\phi\left(\frac{3a+\frac{1}{3}}{1+a}, \frac{1}{3}, \frac{4s}{9}\right) - Bs^{2/3}\phi\left(\frac{1+8a/3}{1+a}, \frac{5}{3}, \frac{4s}{9}\right)\right]$$

$$2u_{y,0} = -Z_{\eta,0} = k^{3/2}B^{3/4}C^{-1/2}[3(a+1)/4]^{1/2}x^{(3a-1)/2} \quad (9)$$

b) $U_e^2 = \sum_{i=0}^{n} A_i x^{r_i}$

The following two conditions will be taken into account: 1) outer velocity not too different from one (e.g., slender bodies, ducts slightly divergent or convergent, etc., and 2) blunt bodies. For case 1, one can assume that the function $x = x(z)$ is equal to the function x_f which applies for the flat plate ($U_e = 1$); namely:

$$x_f = \tfrac{3}{4}[\Gamma(\tfrac{2}{3})(\tfrac{2}{3})^{-1/3}]^{1/2}C^{-1}z^{4/3} = 1.13z^{4/3}$$

whereas for case 2 one can assume that x_s of the stagnation flow ($U_e = kx$) i.e., $x_s = (Ck)^{-1/2}(\frac{9}{4})^{2/3}[\Gamma(\frac{5}{3})/\Gamma(\frac{1}{3})]^{1/2}z^{2/3} = 1.1\ k^{-1/2}z^{2/3}$. After determining in this way the function $f = dx/dz$, we have the first approximation for $x = x(z)$ and can iterate. However, this iteration is not necessary in this case because the error is several percent, and it is acceptable for the purposes of this analysis.

Then Eq. (8) gives for case 1,

$$\frac{1}{2}P^{-2} = u_{y,0} = mx^{-1/2}\left[U_e^2(0) + \sum_{i=1}^{n} m_{ri}A_ix^{ri}\right] \quad (10)$$

where

$$m = \frac{(\frac{2}{3})^{1/4}(\frac{3}{4})^{1/2}}{\Gamma^{3/4}(\frac{2}{3})}\frac{C^{-1/2}}{2} = 0.342$$

$$m_{ri} = \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4r_i}{3}+1\right)\Big/\Gamma\left(\frac{4r_i+1}{3}\right)$$

$m_1 = 3.55$ $m_2 = 5.34$ $m_3 = 6.90$ $m_4 = 8.30$
$m_5 = 9.59$ $m_6 = 10.8$ $m_7 = 11.9$ $m_8 = 13$

For case 2 we have

$$u_{y,0} = \sum_{i=1}^{n} n_{ri}k^{-1/2}A_ix^{r_i-1}$$

where

$$\eta_{ri} = \frac{C^{-1/2}}{2}\left(\frac{9}{4}\right)^{2/3}\Gamma(\tfrac{1}{3})\frac{\Gamma[(2r_i+3)/3]}{\Gamma[(2r_i+1)/3]}\frac{(\frac{2}{3})^{1/3}}{\Gamma(\frac{2}{3})}\left[\frac{\Gamma(\frac{5}{3})}{\Gamma(\frac{1}{3})}\right]^{1/2}$$

$n_1 = 0.85$ $n_2 = 1.25$ $n_3 = 1.59$ $n_4 = 1.90$
$n_5 = 2.18$ $n_6 = 2.45$ $n_7 = 2.70$ $n_8 = 2.95$

6. Accuracy of Results

The accuracy of the method will be proved by a comparison with exact solutions. For class a there are similar solutions of flat plate and of stagnation flow; we have for $u_{y,0}$, $0.332_x{}^{-112}$ and 1.223_x, respectively; the present analysis gives $0.342_T{}^{-1/2}$ and 1.25_T with an error of 3%. For the second class, there are no exact solutions; however some results can be considered sufficiently accurate (see, e.g., Ref. 1).

Table 1 shows a comparison of Howarth's solution for $U_e = 1 - ax$ with that of Kosson.[1] The comparison in Table 2 is more for a circular cylinder (Hiemenz velocity distribution) and sinusinoidal velocity distribution. As can be seen, the error is several percent.

7. Concluding Remarks

A way to evaluate the skin-friction coefficient for two-dimensional incompressible flows has been presented. Of course, the results can be extended easily to three-dimensional axial symmetric flows by means of the Mangler transformation. The results so determined enable us to obtain expression leading to errors of the order of 5%; the accuracy is slightly less than that presented by other methods, but its simplicity is remarkably greater.

References

[1] Kosson, R. L., "An approximate solution for laminar boundary layer flow," AIAA J. 1, 1088–1096 (1963).

[2] Von Kármán, T. and Millikan, C. B., "On the theory of laminar boundary layer involving separation," NACA TR 504 (1934).

[3] Gortler, H., "A new series for the calculation of steady laminar boundary layer flows," Math. Mech. 6, 1–66 (1957).

[4] Tricomi, F. G., *Lezioni sulle Equazioni Integrali* (Gheroni, Torino, Italy, 1958), Chap. I, p. 67.